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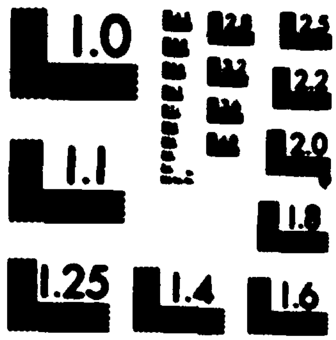
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The Equivalence of Dantzig's Self-Dual Parametric Algorithm for Linear Programs to Lemke's Algorithm for Linear Complementarity Problems Applied to Linear Programs

by

Irvin J. Lustig

TECHNICAL REPORT SOL 87-4

May 1987

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12

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The Equivalence of
Dantzig's Self-Dual Parametric Algorithm for Linear Programs
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ABSTRACT

Dantzig has asserted that his self-dual parametric algorithm for solving a linear program is equivalent to Lemke's method for solving a linear complementarity problem when the latter is applied to solve a linear program. In this paper, we formally prove that Dantzig's assertion is correct—specifically that the point reached along the solution path after $2t$ iterations of Lemke's method is identical with the point reached after t iterations of Dantzig's method.

1. Introduction

Dantzig (1963, Chapter 11) introduced his self-dual parametric algorithm to solve the linear program

$$\text{minimize} \quad c^T x \tag{1.1a}$$

$$\text{subject to} \quad Ax \geq b \tag{1.1b}$$

$$x \geq 0, \tag{1.1c}$$

which is equivalent to solving the linear complementarity problem

$$\begin{pmatrix} -b \\ c \end{pmatrix} + \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.2a}$$

$$u^T y + v^T x = 0, \tag{1.2b}$$

$$u, v, x, y \geq 0. \tag{1.2c}$$

Ravindran (1970) credits Dantzig with the claim that the solution path resulting from the application of the self-dual algorithm to the linear program (1.1) corresponds to the one obtained by applying Lemke's (1965) algorithm to the linear complementarity problem (1.2). Ravindran developed a modified form of Lemke's algorithm which he proved has a solution path that corresponds to the self-dual algorithm. He stated "since our method is a condensed form of Lemke's method in some sense, we have also shown that Dantzig's claim may be valid." However, Ravindran never proved that his method was equivalent to Lemke's algorithm.

McCammon (1970), in his Ph.D. thesis at Rensselaer Polytechnic Institute, considered Dantzig's claim as well. He presented his own parametric pivoting method for general linear complementarity problems of the form

$$q + Mz = w, \tag{1.3a}$$

$$w^T z = 0, \tag{1.3b}$$

$$w, z \geq 0, \tag{1.3c}$$

of which (1.2) is a special case. McCammon* proved that his algorithm is equivalent to Lemke's algorithm applied to (1.3). McCammon, however, believed that the solution path of his algorithm did not correspond to that of Dantzig's self-dual algorithm. In the last section of Appendix II of his thesis, he applied his algorithm and Dantzig's algorithm to a numerical example in order to show that the two solution paths need not correspond. A careful examination of his calculations indicates that he applied the two algorithms to two *different* linear programs.

Section 2 of this paper gives a brief description of each of the algorithms; Section 3 contains a formal proof of Dantzig's claim by showing how each of these algorithms when applied to the same problem follow corresponding solution paths; Section 4 illustrates the proof with a simple example.

2. Algorithm Descriptions

In order to simplify the discussion, we make the non-degeneracy assumption that the linear program (1.1) has no primal or dual degenerate vertices. Otherwise, when a ratio test is performed in each method, the same rules must be used to break ties. The equivalence of these rules is then clear by their construction.

The self-dual algorithm is initialized for the linear program (1.1) by arbitrarily choosing any vectors $f > 0$ and $d > 0$ and scalar $\theta > 0$ sufficiently large so that $-b + \theta f \geq 0$ and $c + \theta d \geq 0$. The parametric linear program

$$\text{minimize} \quad (c + \theta d)^T x \tag{2.1a}$$

$$-Ax + u = -b + \theta f \tag{2.1b}$$

$$x, u \geq 0 \tag{2.1c}$$

has $x = 0$ and $u = -b + \theta f$ as an optimal basic primal feasible solution for all sufficiently large $\theta \geq 0$. The self-dual algorithm lowers θ to some critical value θ_0, such that for a small

* A short description of McCammon's algorithm (without proof) is given by Lemke (1970), pp. 359-361. I find McCammon's proof and the statement of his algorithm somewhat unclear. It is possible that his algorithm is equivalent to Cottle's (1972) parametric pivoting algorithm. Cottle's algorithm is well-defined when M has positive principal minors or M is positive semi-definite. If they are equivalent, then it is not clear whether McCammon realized the necessity of having assumptions on the properties of M in order to make his algorithm well-defined.

$\epsilon > 0$ and $\theta = \theta_0 - \epsilon$, the current solution to (2.1) is either primal or dual infeasible for exactly one primal or dual variable, by the non-degeneracy assumption. If primal (dual) infeasibility occurs, then the dual (primal) simplex method is invoked for one iteration on (2.1) with $\theta = \theta_0$. The theory states that θ can then be lowered to a value less than θ_0 while preserving feasibility. This allows the procedure to be iteratively repeated, stopping at some iteration when $\theta = 0$ is reached or $\theta = \theta* > 0$ is reached below which θ cannot be lowered. In the latter case, (2.1) is either primal or dual infeasible (or both).

Lemke's algorithm is initialized for (1.2) by choosing the same f and d as in the self-dual method and writing

$$\begin{pmatrix} -b \\ c \end{pmatrix} + \begin{pmatrix} f \\ d \end{pmatrix} \bar{\theta} + \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.1a}$$

$$u^T y + v^T x = 0, \tag{2.1b}$$

$$u, v, x, y \geq 0. \tag{2.1c}$$

For all sufficiently large $\bar{\theta}$, $u = -b + \bar{\theta} f > 0$, $v = c + \bar{\theta} d > 0$, $x = y = 0$ is a basic feasible solution to (2.1). The variable pairs (x_j, v_j) and (u_i, y_i) are termed *complementary*. Lemke's algorithm first makes the nonbasic artificial variable $\bar{\theta}$ basic by first decreasing its value from $+\infty$ to some critical value $\bar{\theta} = \bar{\theta}_0$ at which some basic variable becomes 0 and is replaced by $\bar{\theta}$ as a basic variable. $\bar{\theta}$ remains basic from this point on until its value in some subsequent basic solution is reduced to zero. The complement of the variable that exited from the set of basic variables becomes on the next iteration the new entering nonbasic variable. This new variable is made basic by pivoting out some other basic variable, while leaving all the other basic variables nonnegative and maintaining a solution to (2.1). The complement of the variable just pivoted out becomes the new entering nonbasic variable and the procedure is continued in this way, maintaining an almost complementary form until either $\bar{\theta}$ exits the basis at $\bar{\theta} = 0$ or no basic variable blocks the increase of the incoming nonbasic variable, in which case the linear program is either primal or dual infeasible (or both).

It is important to understand that the self-dual algorithm manipulates only the data of the matrix A of the primal linear program while Lemke's algorithm operates both on

the A matrix of the primal and the A^T matrix of the dual problems. Hence, one pivot in the self-dual algorithm which explicitly exchanges the role of a basic and non-basic primal variable implicitly exchanges two corresponding dual variables as well. The set of basic variables for the self-dual method highlights the m primal basic variables and treats the n basic dual variables implicitly. For Lemke's method, the set of basic variables is a collection of some $(m+n-1)$ primal and dual variables of the original problem and $\bar{\theta}$. In the self-dual algorithm, when x_j enters (leaves) the set of basic variables, its complement v_j implicitly leaves (enters) the set of basic variables. Similarly, when u_i leaves (enters) the basic variable set, its complement y_i implicitly enters (leaves) that set. In Lemke's method, all of the components of x, u, v, y are explicitly involved in the pivot process. When a variable (e.g., u_i) leaves the set of basic variables, its complement (e.g., y_i) must enter that set on the next iteration.

3. Proof of Equivalence

To show the equivalence of the self-dual algorithm and Lemke's algorithm applied to the linear programming problem, the algebraic expressions of the terms of the simplex tableaus of each method and their updates are presented in such a way that their correspondence is clear. We assume that $A \in \Re^{m \times n}$, $b \in \Re^m$, and $c \in \Re^n$. Hence the primal vector x is an element of $\Re^n$, and the dual vector y is an element of $\Re^m$. Note that $u \in \Re^m$ and $v \in \Re^n$.

The simplex tableau for the self-dual method is initialized as follows:

	$x_1, \ldots, x_n$	$u_1, \ldots, u_m$	RHS	FBAR
u_1 $\vdots$ u_m	$-A$	I	$-b$	f
z	c	0		
DBAR	d	0		$\theta = +\infty$

Tableau 3.1

The initial basic solution is

$$u = -b + f\theta \quad \Longrightarrow \quad y = 0 \tag{3.1a}$$

$$v = \quad c + d\theta \qquad \Longrightarrow \qquad x = 0 \tag{3.1b}$$

for $\theta = \theta_0$.

The simplex tableau for Lemke's method is initialized in a form that simplifies the algebraic updating:

	$u_1, \ldots, u_m$	$v_1, \ldots, v_n$	$y_1, \ldots, y_m$	$x_1, \ldots, x_n$	$\bar{\theta}$	RHS
u_1 $\vdots$ u_m	I_m	0	0	$-A$	$-f$	$-b$
v_1 $\vdots$ v_n	0	I_n	A^T	0	$-d$	c

Tableau 3.2

The initial basic solution is

$$\left.\begin{array}{l} u = f\bar{\theta} - b \\ v = d\bar{\theta} + c \end{array}\right\} \Longrightarrow \text{nonbasics} \quad \left\{\begin{array}{l} y = 0 \\ x = 0 \end{array}\right. \tag{3.2}$$

for $\bar{\theta} = \theta_0$. The inductive hypothesis for the proof will assume that at iteration t of the self-dual method, and iteration $2t$ of Lemke's algorithm, each method has performed pivots and exchanged labels of complementary pairs of variables to produce Tableaus 3.1 and 3.2, respectively. In fact, the self-dual method tableau will correspond exactly to Tableau 3.1, while Lemke's method will have the variable $\bar{\theta}$ in its set of basic variables and one variable among the u_i and v_j not in the basic set. It will be shown that $\bar{\theta}$ can be pivoted out of the canonical form of Lemke's algorithm at its current value and a non-basic variable pivoted into the basis in such a way that the new tableau is in the form of Tableau 3.2.

Furthermore, the inductive hypothesis will also assume that $-b + f\theta_t \geq 0$ and $c + d\theta_t \geq 0$, where θ_t is the value of θ on iteration t. This inductive assumption allows both methods to restart from their relabeled tableaus.

For initialization, both methods begin with $\theta_0 = +\infty$ and decrease $\theta = \bar{\theta}$ to some blocking value $\theta = \bar{\theta} = \theta_1$. The second part of the inductive hypothesis is satisfied. Because of the symmetry between the primal and dual linear programs, we can assume

that the self-dual method will do a dual-simplex iteration initially, pivoting out some u_i. Similarly, for Lemke's method, we can assume that $\bar{\theta}$ will enter into the set of basic variables, replacing some primal variable u_i. Therefore, just before pivoting u_i out of the basis, the canonical form of Tableau 3.1 and Tableau 3.2 correspond and thus, the test

$$\min\left\{\min_{i=1,\ldots,m}\left\{\frac{-b_i}{f_i} : f_i > 0\right\}, \min_{j=1,\ldots,n}\left\{\frac{c_j}{d_j} : d_j > 0\right\}\right\} \tag{3.3}$$

performed by Lemke's Method is seen to be exactly the same as the self-dual test and therefore the u_i chosen to exit will be the same. Note in Tableau 3.2, $\bar{\theta}$ is decreasing from $+\infty$.

For both methods, let i_1 correspond to some primal slack variable u_{i_1} which will exit the basis. By reordering the equations of the primal linear program, we can assume that $i_1 = 1$. The self-dual method will set $\theta = b_1/f_1$, and Lemke's method will perform a pivot in the column corresponding to $\bar{\theta}$ and the row corresponding to u_1. This pivot will cause $\bar{\theta}$ to be made basic, replacing u_1, and will set $\bar{\theta} = -b_1/(-f_1) = \theta$.

The self-dual method will now perform a dual-simplex ratio test, using θ to determine the parametric reduced costs $c + d\theta$, when $\theta = b_1/f_1$:

$$\min_{j=1,\ldots,n}\left\{\frac{c_j + d_j\theta}{a_{1j}} : -a_{1j} < 0\right\} = \min_{j=1,\ldots,n}\left\{\frac{f_1c_j + d_jb_1}{f_1a_{1j}} : a_{1j} > 0\right\} \tag{3.4}$$

Lemke's method pivots u_1 out of the basis, and its complement, y_1, is required to be the entering variable. The components of RHS for Lemke's method are, when $\bar{\theta} = b_1/f_1$:

$$\mathrm{RHS}(k) = \begin{cases} b_1/f_1 & \text{if } k = 1; \\ -b_k + f_k(b_1/f_1) & \text{if } k = 2,\ldots,m; \\ c_j + d_j(b_1/f_1) & \text{if } k = m+1,\ldots,m+n;\ j = k-m. \end{cases} \tag{3.5}$$

Because the coefficient in Tableau 3.2 in the (u_1, y_1) position is zero, the column for y_1 in Tableau 3.2 remains unchanged after the pivot in the $(u_1, \bar{\theta})$ position. Hence the ratio test on the second iteration of Lemke's algorithm is only made among the variables v_j, $j = 1,\ldots,n$ and is therefore

$$\min_{j=1,\ldots,n}\left\{\frac{c_j + d_j(b_1/f_1)}{a_{1j}} : a_{1j} > 0\right\} = \min_{j=1,\ldots,n}\left\{\frac{f_1c_j + d_jb_1}{f_1a_{1j}} : a_{1j} > 0\right\}. \tag{3.6}$$

This is the same test as the test (3.4) done in the self-dual method. By reordering the primal variables, we can assume that $j = 2$ is chosen in these ratio tests.

The pivot for the self-dual method is done in the (u_1, x_2) position. If we let E_2 be an elementary matrix, i.e. an identity matrix except for its first column $\bar{E}_1$:

$$\bar{E}_1 = \frac{1}{a_{12}} \begin{pmatrix} -1 \\ -a_{22} \\ \vdots \\ -a_{m2} \end{pmatrix} \quad \text{and} \quad E_1 = \begin{pmatrix} \bar{E}_1 & \begin{matrix} 0 \\ I_{m-1} \end{matrix} \end{pmatrix},$$

where I_p is a $p{\times}p$ identity matrix, the new tableau is:

	$x_1, \ldots, x_n$	$u_1, \ldots, u_m$	RHS	FBAR
x_2, u_2, $\vdots$, u_m	$-E_1 A$	E_1	$-E_1 b$	$E_1 f$
z	$(-c_2/a_{12})a_{1\cdot} + c$	(c_2/a_{12}) 0		
DBAR	$(-d_2/a_{12})a_{1\cdot} + d$	(d_2/a_{12}) 0		$\theta = b_1/f_1$

Tableau 3.3

where $a_{1\cdot}$ denotes the first row of A. While the self-dual method has done a pivot on the primal linear program, it implicitly has done a pivot on the dual linear program. The method always maintains complementarity between the primal and dual solutions in the tableau. Thus, at the same time that x_2 entered the set of basic variables, replacing u_1, y_1 implicitly replaced v_2. After two pivots in Lemke's method, $\bar{\theta}$ has replaced u_1, and y_1 has replaced v_2. x_2 is now the variable entering into the set of basic variables. We will now show that the Lemke system can be pivoted in such a way that x_2 replaces $\bar{\theta}$, and the bases between the two methods will correspond again after this pivot. Later we will show that Lemke's method uses x_2 as the driving variable to choose the same outgoing variable as the self-dual method, while $\bar{\theta}$ remains in the basis the entire time (until optimality). It is now time for the first lemma.

Lemma 3.1. *In Lemke's method, after two pivots, the coefficient in the x_2 column for $\bar{\theta}$ is positive. In other words, $d\bar{\theta}/dx_2 < 0$.*

Proof. After the first pivot, the desired coefficient $\alpha = -a_{12}/(-f_1) = a_{12}/f_1$. Since the coefficient in the first row in the column for y_1 is zero, the second pivot does not modify α. Hence α remains positive since $a_{12} > 0$ and $f_1 > 0$ from the previous ratio tests (3.3) and (3.6). ∎

If after doing the first two pivots of Lemke's method, we pivot on this nonzero element α, we form a new tableau that can be obtained by doing a block pivot

$$\begin{array}{cc} & \begin{array}{cc} y_1 & x_2 \end{array} \\ \begin{array}{c} u_1 \\ v_2 \end{array} & \begin{pmatrix} 0 & -a_{12} \\ a_{12} & 0 \end{pmatrix} \end{array}$$

on Tableau 3.1. This is the same as performing the exchange $\langle v_2, y_1 \rangle$ followed by $\langle u_1, x_2 \rangle$. This is done by setting

$$\bar{E}_2 = \frac{1}{a_{12}} \begin{pmatrix} -a_{11} \\ 1 \\ -a_{13} \\ \vdots \\ -a_{1n} \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & & 0 \\ 0 & \bar{E}_2 & 0 \\ \vdots & & \\ 0 & & I_{n-2} \end{pmatrix} \tag{3.7}$$

and creating the following tableau:

	$u_1, \ldots, u_m$	$v_1, \ldots, v_n$	$y_1, \ldots, y_m$	$x_1, \ldots, x_n$	$\bar{\theta}$	RHS
$u_1, \ldots, u_m$	I_m	0	0	$-A$	$-f$	$-b$
$v_1, y_1, v_3, \ldots, v_n$	0	E_2	$E_2 A^T$	0	$-E_2 d$	$E_2 c$

Tableau 3.4

Then we exchange u_1 and x_2 by pivoting on the $\langle u_1, x_2 \rangle$ element. This pivot is very

similar to the pivot done in the self-dual method. It produces the following system:

	$u_1, \dots, u_m$	$v_1, \dots, v_n$	$y_1, \dots, y_m$	$x_1, \dots, x_n$	$\bar{\theta}$	RHS
x_2 u_2 $\vdots$ u_m	E_1	0	0	$-E_1 A$	$-E_1 f$	$-E_1 b$
v_1 y_1 v_3 $\vdots$ v_n	0	E_2	$E_2 A^T$	0	$-E_2 d$	$E_2 c$

Tableau 3.5

This is a valid canonical form for Lemke's method, since at $\bar{\theta} = \theta_1$, the system is feasible. Lemke's method can be started from Tableau 3.5 with $\bar{\theta}$ as the first entering variable. By examining Tableaus 3.3 and 3.5, the ratio tests to find the primal variable blocking the decrease of θ, i.e.,

$$\min_{i=1,\dots,m} \{(-E_1 b)_i/(E_1 f)_i : (E_1 f)_i > 0\}, \tag{3.8}$$

are seen to be the same for both methods. Note that $(E_2 c)_j = c_j - (a_{1j}/a_{12})c_2$ for $j \neq 2$, $(E_2 c)_2 = c_2/a_{12}$, $(-E_2 d)_j = -d_j + (a_{1j}/a_{12})d_2$ for $j \neq 2$, and $(-E_2 d)_2 = -d_2/a_{12}$. This corresponds to the rows CBAR and DBAR in Tableau 3.3. Hence the ratio test to find the dual variable blocking the decrease of θ,

$$\min_{j=1,\dots,n} \{(E_2 c)_j/(E_2 d)_j : (E_2 d)_j > 0\}, \tag{3.9}$$

is the same for both methods. We have now proved the following:

Lemma 3.2. *After one pivot in the self-dual method and two pivots in Lemke's method, the driving variable for Lemke's method can be pivoted into the basis for $\bar{\theta}$ such that the canonical forms of the new systems after rearrangement of rows and columns correspond to Tableaus 3.1 and 3.2. Each method can be restarted from its respective new tableau.*

The correspondence in Lemma 3.2 is achieved by relabeling the variables in Tableaus 3.3 and 3.5. This is done by letting the m primal basic variables represent u. If this is

done, then the two tableaus correspond to the same linear program, which is in a new canonical form. The new form is equivalent to the initial form of the linear program by pivoting.

However, Lemke's method has a driving variable which does not replace $\bar{\theta}$. We will now compare the ratio tests done by the self-dual method after one pivot and Lemke's method after 2 pivots.

The self-dual method performs the following ratio test on the primal variables:

$$\begin{aligned}
&\min_{i=2,\ldots,m}\left\{\frac{(-E_1 b)_i}{(E_1 f)_i} : (E_1 f)_i > 0\right\} \\
&= \min_{i=2,\ldots,m}\left\{\frac{-(b_i - (a_{i2}/a_{12})b_1)}{f_i - (a_{i2}/a_{12})f_1} : f_i - (a_{i2}/a_{12})f_1 > 0\right\} \\
&= \min_{i=2,\ldots,m}\left\{\frac{a_{i2}b_1 - b_i a_{12}}{a_{12}f_i - a_{i2}f_1} : a_{12}f_i - a_{i2}f_1 > 0\right\}
\end{aligned} \tag{3.10}$$

When $i = 1$,

$$\frac{(-E_1 b)_i}{(E_1 f)_i} = \frac{b_1/a_{12}}{-f_1/a_{12}}. \tag{3.11}$$

Since the ratio test is only considered when the denominator is positive, and $f_1 > 0$ and $a_{12} > 0$ from the previous iteration, $-f_1/a_{12} < 0$, which implies that $i = 1$ has no effect on this ratio test.

The self-dual method performs the following ratio test on the dual variables:

$$\min_{j=1,\ldots,n}\left\{\frac{c_j - (a_{1j}/a_{12})c_2}{d_j - (a_{1j}/a_{12})d_2} : d_j - (a_{1j}/a_{12})d_2 > 0\right\} \tag{3.12}$$

$$= \min_{j=1,\ldots,n}\left\{\frac{a_{12}c_j - c_2 a_{ij}}{a_{12}d_j - a_{1j}d_2} : a_{12}d_j - a_{1j}d_2 > 0\right\} \tag{3.13}$$

The dual variable for u_1 is also included in the ratio test, when $d_2 > 0$, yielding the ratio

$$\frac{c_2/a_{12}}{d_2/a_{12}} = \frac{c_2}{d_2}. \tag{3.14}$$

For Lemke's method, let

$$\bar{E}_3 = \frac{-1}{f_1}\begin{pmatrix} 1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}, \quad E_3 = \begin{pmatrix} \bar{E}_3 & \begin{matrix} 0 \\ I_{m-1} \end{matrix} \end{pmatrix}, \tag{3.15}$$

$$\bar{E}_4 = \frac{-1}{f_1}\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}, \quad \text{and} \quad E_4 = (\,\bar{E}_4 \quad I_n\,). \tag{3.16}$$

Then, after one pivot on Tableau 3.2, the following tableau is obtained.

	$u_1,\ldots,u_m$	$v_1,\ldots,v_n$	$y_1,\ldots,y_m$	$x_1,\ldots,x_n$	$\bar{\theta}$	RHS
$\bar{\theta}$					1	
$u_2, \ldots, u_m$	E_3	0	0	$-E_3A$	0 $\vdots$	$-E_3b$
$v_1, \ldots, v_n$	$\bar{E}_4$ $\;0$	I_n	A^T	$-\bar{E}_4 a_{1\cdot}^T$	$\vdots$ 0	$E_4\begin{pmatrix} -b_1 \\ c \end{pmatrix}$

Tableau 3.6

y_1 is now the driving variable. The next pivot is on the (v_2, y_1) element. It produces the tableau

	$u_1,\ldots,u_m$	$v_1,\ldots,v_n$	$y_1,\ldots,y_m$	$x_1,\ldots,x_n$	$\bar{\theta}$	RHS
$\bar{\theta}$					1	
$u_2, \ldots, u_m$	E_3	0	0	$-E_3A$	0 $\vdots$	$-E_3b$
$v_1, y_1, v_3, \ldots, v_n$	$E_2\bar{E}_4$ $\;0$	E_2	E_2A^T	$-E_2\bar{E}_4 a_{1\cdot}^T$	$\vdots$ 0	$E_2E_4\begin{pmatrix} -b_1 \\ c \end{pmatrix}$

Tableau 3.7

At this point, x_2 is the driving variable. The column α corresponding to x_2 has

$$\alpha_k = \begin{cases} a_{12}/f_1, & \text{if } k = 1; \\ (f_k/f_1)a_{12} - a_{k2}, & \text{if } k = 2,\ldots,m; \\ d_2/f_1, & \text{if } k = m+2; \\ (d_j/f_1)a_{12} - (d_2/f_1)a_{1j}, & \text{if } k = m+1,\ldots,m+n,\ k \neq m+2,\ j = k-m. \end{cases} \tag{3.17}$$

For the right-hand side, the k^{th} element is:

$$\mathrm{RHS}(k) = \begin{cases} b_1/f_1, & \text{if } k = 1; \\ (f_k/f_1)b_1 - b_k, & \text{if } k = 2,\ldots,m; \\ (c_2 + (d_2/f_1)b_1)/a_{12}, & \text{if } k = m+2; \\ c_j + (d_j/f_1)b_1 - (a_{1k}/a_{12})(c_2 + (d_2/f_1)b_1), & \text{if } k = m+1,\ldots,m+n, \\ & \quad k \neq m+2,\ j = k-m. \end{cases} \tag{3.18}$$

The ratio test for the third iteration of Lemke's method will be

$$\min_{k=1,\ldots,m+n} \{\mathrm{RHS}(k)/\alpha_k : \alpha_k > 0\}. \tag{3.19}$$

We now wish to compare the results of the self-dual ratio tests (3.10),(3.13), and (3.14) after one pivot, and the ratio test (3.19) for Lemke's method after two pivots. The term "winner" will refer to the index of the variable (or the variable itself) that is the argument of a minimum ratio test. We will do the following:

(I) Compare the winners among $k = 2,\ldots,m$ in (3.19) and (3.10). Note that $k = 1$ corresponds to termination of Lemke's method.

(II) Compare the winners among $k = m+1,\ldots,n$ in (3.19) and (3.13) or (3.14).

(III) Compare the two winners in (I) and (II).

Lemma 3.3. *Without loss of generality, suppose $i = 4$ is the winner of the self-dual ratio test (3.10) for $i = 2,\ldots,m$. Then $k = 4$ is the winner of the Lemke ratio test (3.19) for $k = 2,\ldots,m$.*

Proof. Suppose on the contrary $k = 3$ was the winner of the Lemke ratio test for $k = 2,\ldots,m$. Then

$$\frac{(f_3/f_1)b_1 - b_3}{(f_3/f_1)a_{12} - a_{32}} < \frac{(f_4/f_1)b_1 - b_4}{(f_4/f_1)a_{12} - a_{42}}, \tag{3.20}$$

which implies

$$\frac{f_3b_1 - f_1b_3}{a_{12}f_3 - a_{32}f_1} < \frac{f_4b_1 - f_1b_4}{a_{12}f_4 - a_{42}f_1}, \tag{3.21}$$

with $(f_3/f_1)a_{12} - a_{32} > 0$ and $(f_4/f_1)a_{12} - a_{42} > 0$ (by equation (3.19)). Since $i = 4$ was the winner of the self-dual ratio test (3.10) for $i = 2,\ldots,m$,

$$\frac{a_{42}b_1 - b_4a_{12}}{a_{12}f_4 - a_{42}f_1} < \frac{a_{32}b_1 - b_3a_{12}}{a_{12}f_3 - a_{32}f_1}. \tag{3.22}$$

Note that the Lemke ratio test implies that the denominators agree between the two methods. The difference between the numerators would lead one to believe that the two tests are not equivalent.

Equation (3.22) implies (after cross multiplication)

$$a_{12}a_{42}b_1f_3 - a_{12}^2f_3b_4 - a_{32}a_{42}b_1f_1 + a_{12}a_{32}f_1b_4 < \\ a_{12}a_{32}b_1f_4 - a_{12}^2f_4b_3 - a_{32}a_{42}b_1f_1 + a_{12}a_{42}f_1b_3, \quad (3.23)$$

which implies

$$a_{12}b_1(a_{42}f_3 - a_{32}f_4) - a_{12}^2(f_3b_4 - b_3f_4) + a_{12}f_1(a_{32}b_4 - a_{42}b_3) < 0, \quad (3.24)$$

which implies

$$b_1(a_{42}f_3 - a_{32}f_4) - a_{12}(f_3b_4 - b_3f_4) + f_1(a_{32}b_4 - a_{42}b_3) < 0. \quad (3.25)$$

Equation (3.21) implies (after cross multiplication)

$$a_{12}b_1f_3f_4 - a_{12}f_1b_3f_4 - a_{42}b_1f_1f_3 + a_{42}f_1^2b_3 < \\ a_{12}b_1f_3f_4 - a_{12}f_1f_3b_4 - a_{32}f_1b_1f_4 + a_{32}f_1^2b_4, \quad (3.26)$$

which implies

$$0 < f_1b_1(a_{42}f_3 - a_{32}f_4) - f_1^2(a_{42}b_3 - a_{32}b_4) + a_{12}f_1(b_3f_4 - f_3b_4), \quad (3.27)$$

which implies

$$0 < b_1(a_{42}f_3 - a_{32}f_4) - a_{12}(f_3b_4 - b_3f_4) + f_1(a_{32}b_4 - a_{42}b_3). \quad (3.28)$$

Statements (3.25) and (3.28) are contradictory. Hence the two ratio tests must declare the same winner.∎

Lemma 3.4. *Without loss of generality, suppose $j = 4$ is the winner of the self-dual ratio tests (3.13) and (3.14) for $j = 1, \ldots, n$. Then $k = m + 4$ is the winner of the Lemke ratio test (3.19) for $k = m + 1, \ldots, m + n$.*

Proof. Throughout this proof, the relation $k = m+j$ will hold. Hence, we can interchange the use of j and k througout the proof. Suppose $j = 4$ is not the winner of the Lemke

ratio test. There are two cases to consider, depending upon whether the winner j is $j = 2$ or $j \neq 2$, which will correspond to using equations (3.14) and (3.13) respectively.

Case 1: $j \neq 2$ is the winner. Without loss of generality, assume $j = 3$ is the winner. The Lemke ratio test (3.19) implies:

$$\frac{c_3 - \frac{a_{13}}{a_{12}}c_2 + \frac{b_1}{f_1}(d_3 - \frac{a_{13}}{a_{12}}d_2)}{\frac{d_3}{f_1}a_{12} - \frac{d_2}{f_1}a_{13}} < \frac{c_4 - \frac{a_{14}}{a_{12}}c_2 + \frac{b_1}{f_1}(d_4 - \frac{a_{14}}{a_{12}}d_2)}{\frac{d_4}{f_1}a_{12} - \frac{d_2}{f_1}a_{14}}, \tag{3.29}$$

which implies

$$\frac{c_3 f_1 - \frac{a_{13}}{a_{12}}c_2 f_1 + \frac{b_1}{a_{12}}(d_3 a_{12} - a_{13}d_2)}{d_3 a_{12} - d_2 a_{13}} < \frac{c_4 f_1 - \frac{a_{14}}{a_{12}}c_2 f_1 + \frac{b_1}{a_{12}}(d_4 a_{12} - a_{14}d_2)}{d_4 a_{12} - d_2 a_{14}}, \tag{3.30}$$

which implies

$$\frac{c_3 - \frac{a_{13}}{a_{12}}c_2}{d_3 a_{12} - d_2 a_{13}} < \frac{c_4 - \frac{a_{14}}{a_{12}}c_2}{d_4 a_{12} - d_2 a_{14}} \tag{3.31}$$

Since $j = 4$ was the winner of the self-dual ratio test (3.13),

$$\frac{a_{12}c_4 - c_2 a_{14}}{a_{12}d_4 - a_{14}d_2} < \frac{a_{12}c_3 - c_2 a_{13}}{a_{12}d_3 - a_{13}d_2}. \tag{3.32}$$

Since $a_{12} > 0$, equation (3.32) contradicts equation (3.31). So, for the first case, $k = m + 4$ must be the winner of the Lemke ratio test (3.19).

Case 2: $k = m + 2$ is the winner of the Lemke test (3.19). Then $d_2 > 0$. The ratio test implies

$$\frac{(c_2 + (d_2/f_1)b_1)/a_{12})}{d_2/f_1} < \frac{c_4 - (a_{14}/a_{12})c_2 + (b_1/f_1)(d_4 - (a_{14}/a_{12})d_2)}{(d_4/f_1)a_{12} - (d_2/f_1)a_{14}}, \tag{3.33}$$

which implies

$$\frac{c_2 f_1}{d_2 a_{12}} < \frac{c_4 f_1 - (a_{14}/a_{12})c_2 f_1}{d_4 a_{12} - d_2 a_{14}}, \tag{3.34}$$

which implies

$$\frac{c_2}{d_2} < \frac{a_{12}c_4 - c_2 a_{14}}{a_{12}d_4 - a_{14}d_2}. \tag{3.35}$$

The self-dual ratio tests (3.13) and (3.14) imply:

$$\frac{a_{12}c_4 - c_2 a_{14}}{a_{12}d_4 - a_{14}d_2} < \frac{c_2}{d_2}. \tag{3.36}$$

Statements (3.35) and (3.36) are contradictory. Hence $k = m + 4$ must be the winner of the Lemke test. ∎

Corollary. *If the dual variable for u_1 wins the self-dual ratio test (3.13) and (3.14) among the dual variables, then $k = m + 2$ will win the Lemke ratio test.*

Proof. Similar to Lemma 3.4, Case 2.∎

We will now compare the winners of the primal and dual ratio tests.

Lemma 3.5. *Without loss of generality, assume $i = 3$ is the overall winner of the self-dual ratio tests (3.10), (3.13) and (3.14) between the primal and the dual variables. Then $k = 3$ is the overall winner in the Lemke ratio test (3.19) for $k = 1, \ldots, m + n$.*

Proof. Without loss of generality, assume that $i = 3$ and $j = 4$ are the winners among the primal and dual variables, respectively, in the self-dual ratio tests. By Lemmas 3.3 and 3.4, $k = 3$ and $k = m + 4$ are the winners for the two Lemke ratio tests (3.19) among $k = 1, \ldots, m$ and $k = m + 1, \ldots, m + n$, respectively. For the self-dual ratio test, the ratio for $i = 3$ must be less than the ratio for $j = 4$. On the contrary, if $k = 3$ is not the overall winner for the Lemke test, then $k = m + 4$ must be. The Lemke test (3.19) then implies

$$\frac{c_4 - \frac{a_{14}}{a_{12}}c_2 + \frac{b_1}{f_1}(d_4 - \frac{a_{14}}{a_{12}}d_2)}{\frac{d_4}{f_1}a_{12} - \frac{d_2}{f_1}a_{14}} < \frac{\frac{f_3}{f_1}b_1 - b_3}{\frac{f_3}{f_1}a_{12} - a_{32}}, \tag{3.37}$$

which implies

$$\frac{f_1c_4 - \frac{a_{14}}{a_{12}}f_1c_2}{d_4a_{12} - d_2a_{14}} + \frac{b_1}{a_{12}} < \frac{f_3b_1 - b_3f_1}{a_{12}f_3 - a_{32}f_1}, \tag{3.38}$$

which implies

$$\frac{a_{12}f_1c_4 - a_{14}f_1c_2}{d_4a_{12} - d_2a_{14}} < \frac{a_{12}f_3b_1 - b_3f_1a_{12} - b_1(a_{12}f_3 - a_{32}f_1)}{a_{12}f_3 - a_{32}f_1}, \tag{3.39}$$

which implies

$$\frac{f_1(a_{12}c_4 - a_{14}c_2)}{d_4a_{12} - d_2a_{14}} < \frac{f_1(a_{32}b_1 - b_3a_{12})}{a_{12}f_3 - a_{32}f_1}. \tag{3.40}$$

The self-dual ratio test implies

$$\frac{a_{32}b_1 - b_3a_{12}}{a_{12}f_3 - a_{32}f_1} < \frac{a_{12}c_4 - a_{14}c_2}{d_4a_{12} - d_2a_{14}}. \tag{3.41}$$

Since $f_1 > 0$, (3.40) and (3.41) are contradictory. Hence $k = 3$ must be the overall winner.∎

Lemma 3.6. *If $j = 4$ is the overall winner of the self-dual ratio tests (3.10) and (3.13), then $k = m + 4$ is the overall winner of the Lemke ratio test (3.19).*

Proof. Similiar to Lemma 3.5.∎

Lemma 3.7. *If the dual variable on u_1 is the overall winner of the self-dual ratio tests (3.10), (3.13) and (3.14), then $k = m + 2$ is the overall winner of the Lemke ratio test (3.19).*

Proof. Suppose on the contrary $i = 3$ is the winner among the primal variables of the self-dual ratio test (3.10), and that u_1 is the winner among all the dual variables. By Lemmas 3.3 and 3.4, $k = 3$ and $k = m + 2$ are the winners in the Lemke ratio test (3.19) among $k = 1, \ldots, m$ and $k = m + 1, \ldots, m + n$, respectively. The self-dual ratio test implies:

$$\frac{c_2}{d_2} < \frac{a_{32}b_1 - b_3a_{12}}{a_{12}f_3 - a_{32}f_1}. \tag{3.42}$$

The Lemke ratio test implies

$$\frac{f_3b_1 - f_1b_3}{a_{12}f_3 - a_{32}f_1} < \frac{(c_2 + d_2\frac{b_1}{f_1})/a_{12}}{d_2/f_1}, \tag{3.43}$$

which implies

$$\frac{-b_1}{a_{12}} + \frac{f_3b_1 - f_1b_3}{a_{12}f_3 - a_{32}f_1} < \frac{c_2f_1}{a_{12}d_2}, \tag{3.44}$$

which implies

$$\frac{-b_1(a_{12}f_3 - a_{32}f_1) + f_3b_1a_{12} - f_1b_3a_{12}}{a_{12}f_3 - a_{32}f_1} < \frac{c_2f_1}{d_2}, \tag{3.45}$$

which implies

$$\frac{a_{32}f_1b_1 - f_1b_3a_{12}}{a_{12}f_3 - a_{32}f_1} < \frac{c_2f_1}{d_2}. \tag{3.46}$$

Since $f_1 > 0$, (3.42) and (3.46) are contradictory. Hence $k = m + 2$ must be the overall winner of the Lemke ratio test.∎

We have now shown the following: If Lemke's method is executed in a normal manner, x_2 will become the driving variable on the third iteration. The same variable will block x_2 as if x_2 replaced $\bar{\theta}$, and then Lemke's method was restarted from the new skew-symmetric form, with a ratio test done between the column RHS and $\bar{\theta}$. But this form is equivalent to restarting the self-dual method after one iteration. If the self-dual method takes ℓ iterations, this argument can be used for each $t = 1, 2, \ldots\ \ell$, since the inductive assumptions will hold on each of these iterations. Hence we have proved:

Theorem. *Assume that Dantzig's self-dual parametric algorithm is executed on (1.1) and that Lemke's algorithm is executed on (1.2) and an optimal solution is found by the self-dual method in ℓ iterations. Then the pivots of iteration t of the self-dual algorithm correspond in a precise way to the pivots of iterations $2t-1$, $2t$ and $2t+1$ of Lemke's algorithm, for $t = 1, 2, \ldots, \ell$.*

Note that Lemke's method will take $2\ell + 1$ iterations. The extra pivot comes when finding the optimal solution, when the last driving variable replaces $\bar{\theta}$.

4. An Example

In this section, we give an example to illustrate the theorem. In the tableaus the angle brackets $\langle\rangle$ indicate the next pivot that will occur. Consider the linear program

$$\begin{aligned} \text{minimize} \quad & 4x_1 - 3x_2 \\ \text{subject to} \quad & -2x_1 - 2x_2 \geq -19 \\ & x_1 - x_2 \geq 1 \\ & -x_1 + 2x_2 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The initial tableau corresponding to Tableau 3.1 for the self-dual method is set up as follows:

	x_1	x_2	u_1	u_2	u_3	RHS	FBAR
u_1	2	2	1	0	0	19	0
u_2	-1	$\langle 1 \rangle$	0	1	0	-1	1
u_3	1	-2	0	0	1	-2	1
z	4	-3	0	0	0		
DBAR	0	1	0	0	0		$\theta = +\infty$

Tableau 4.1

θ is decreased from $+\infty$ and is blocked by the reduced cost for x_2 at a value $\theta_1 = 3$. x_2 is the incoming nonbasic variable and the ratio test indicates that u_2 is the variable that x_2

will replace. The pivot is performed, yielding the tableau

	x_1	x_2	u_1	u_2	u_3	RHS	FBAR
u_1	4	0	1	−2	0	21	−2
x_2	−1	1	0	1	0	−1	1
u_3	$\langle -1 \rangle$	0	0	2	1	−4	3
z	1	0	0	3	0		
DBAR	1	0	0	−1	0		$\theta = 3$

Tableau 4.2

After interchanging the columns for u_2 and x_2 of this tableau, we arrive at a tableau that corresponds to Tableau 4.1:

	x_1	u_2	u_1	x_2	u_3	RHS	FBAR
u_1	4	−2	1	0	0	21	−2
x_2	−1	1	0	1	0	−1	1
u_3	$\langle -1 \rangle$	2	0	0	1	−4	3
z	1	3	0	0	0		
DBAR	1	−1	0	0	0		$\theta = 3$

Tableau 4.2a

At this point, all the reduced costs are positive for $0 \leq \theta \leq 3$. Hence, θ is reduced to a value of $\frac{4}{3}$, since it is blocked by the primal basic variable u_3. The dual simplex method is invoked, and x_1 replaces u_3 in the set of basic variables. The pivot is performed on Tableau 4.2, yielding the tableau:

	x_1	x_2	u_1	u_2	u_3	RHS	FBAR
u_1	0	0	1	6	4	5	10
x_2	0	1	0	−1	−1	3	−2
x_1	1	0	0	−2	−1	4	−3
z	0	0	0	5	1		
DBAR	0	0	0	1	1		$\theta = \frac{4}{3}$

Tableau 4.3

Note that all the reduced costs and primal variables are nonnegative for $\theta = 0$, and hence the optimal solution has been found. This table can be rearranged by interchanging the

columns to yield a tableau that corresponds to Tableau 4.1:

	u_2	u_3	u_1	x_2	x_1	RHS	FBAR
u_1	6	4	1	0	0	5	10
x_2	−1	−1	0	1	0	3	−2
x_1	−2	−1	0	0	1	4	−3
z	5	1	0	0	0		
DBAR	1	1	0	0	0		$\theta = \frac{4}{3}$

Tableau 4.3a

For Lemke's method, the initial tableau is:

	u_1	u_2	u_3	v_1	v_2	y_1	y_2	y_3	x_1	x_2	$\bar{\theta}$	RHS
u_1	1	0	0	0	0	0	0	0	2	2	0	19
u_2	0	1	0	0	0	0	0	0	−1	1	−1	−1
u_3	0	0	1	0	0	0	0	0	1	−2	−1	−2
v_1	0	0	0	1	0	−2	1	−1	0	0	0	4
v_2	0	0	0	0	1	−2	−1	2	0	0	⟨−1⟩	−3

Tableau 4.4

On the first iteration, $\bar{\theta}$ is the entering basic variable. It replaces v_2 in the set of basic variables and produces the tableau:

	u_1	u_2	u_3	v_1	v_2	y_1	y_2	y_3	x_1	x_2	$\bar{\theta}$	RHS
u_1	1	0	0	0	0	0	0	0	2	2	0	19
u_2	0	1	0	0	−1	2	1	−2	−1	⟨1⟩	0	2
u_3	0	0	1	0	−1	2	1	−2	1	−2	0	1
v_1	0	0	0	1	0	−2	1	−1	0	0	0	4
$\bar{\theta}$	0	0	0	0	−1	2	1	−2	0	0	1	3

Tableau 4.5

The complement of v_2, namely x_2, is now the entering basic variable. It replaces u_2 in the set of basic variables and produces the tableau:

	u_1	u_2	u_3	v_1	v_2	y_1	y_2	y_3	x_1	x_2	$\bar{\theta}$	RHS
u_1	1	−2	0	0	2	−4	−2	4	4	0	0	15
x_2	0	1	0	0	−1	2	1	−2	−1	1	0	2
u_3	0	2	1	0	−3	6	⟨3⟩	−6	−1	0	0	5
v_1	0	0	0	1	0	−2	1	−1	0	0	0	4
$\bar{\theta}$	0	0	0	0	−1	2	1	−2	0	0	1	3

Tableau 4.6

The complement of u_2, y_2, is now the entering basic variable. To show the correspondence to Tableau 4.2a, we pivot y_2 into the basis for $\bar{\theta}$, rearrange the columns, and produce the tableau:

	u_1	x_2	u_3	v_1	y_2	y_1	v_2	y_3	x_1	u_2	$\bar{\theta}$	RHS
u_1	1	0	0	0	0	0	0	0	4	−2	2	21
x_2	0	1	0	0	0	0	0	0	−1	1	−1	−1
u_3	0	0	1	0	0	0	0	0	−1	2	−3	−4
v_1	0	0	0	1	0	−4	1	1	0	0	−1	1
y_2	0	0	0	0	1	2	−1	−2	0	0	1	3

Tableau 4.6a

y_2 is the driving variable in Tableau 4.6 and replaces u_3 in the set of basic variables to produce the tableau:

	u_1	u_2	u_3	v_1	v_2	y_1	y_2	y_3	x_1	x_2	$\bar{\theta}$	RHS
u_1	1	−2/3	2/3	0	0	0	0	0	10/3	0	0	55/3
x_2	0	1/3	−1/3	0	0	0	0	0	−2/3	1	0	1/3
y_2	0	2/3	1/3	0	−1	2	1	−2	−1/3	0	0	5/3
v_1	0	−2/3	−1/3	1	1	−4	0	1	1/3	0	0	8/3
$\bar{\theta}$	0	−2/3	−1/3	0	0	0	0	0	1/3	0	1	4/3

Tableau 4.7

The complement of u_3, y_3, is now the entering basic variable. It replaces v_1 in the set of basic variables and produces the tableau:

	u_1	u_2	u_3	v_1	v_2	y_1	y_2	y_3	x_1	x_2	$\bar{\theta}$	RHS
u_1	1	−2/3	2/3	0	0	0	0	0	10/3	0	0	55/3
x_2	0	1/3	−1/3	0	0	0	0	0	−2/3	1	0	1/3
y_2	0	−2/3	−1/3	2	1	−6	1	0	1/3	0	0	19/3
y_3	0	−2/3	−1/3	1	1	−4	0	1	1/3	0	0	8/3
$\bar{\theta}$	0	−2/3	−1/3	0	0	0	0	0	1/3	0	1	4/3

Tableau 4.8

The complement of v_1, x_1, is now the entering basic variable. It replaces $\bar{\theta}$ in the set of

basic variables and produces the optimal tableau:

	u_1	u_2	u_3	v_1	v_2	y_1	y_2	y_3	x_1	x_2	$\bar{\theta}$	RHS
u_1	1	6	4	0	0	0	0	0	0	0	−10	5
x_2	0	−1	−1	0	0	0	0	0	0	1	2	3
y_2	0	0	0	2	1	−6	1	0	0	0	−1	5
y_3	0	0	0	1	1	−4	0	1	0	0	−1	1
x_1	0	−2	−1	0	0	0	0	0	1	0	3	4

Tableau 4.9

By reordering the columns of this tableau, we get a tableau that corresponds to Tableau 4.3a.

The following chart compares the pivots done in each method. For each pivot, the notation (a, b) indicates that b replaces a in the set of basic variables by an explicit pivot. For the self-dual method, the notation $\langle a, b\rangle$ indicates that b replaces a by an implicit pivot.

<table>
<tr><th colspan="2">Self-Dual</th><th colspan="2">Lemke</th></tr>
<tr><th>Itn</th><th>Pivot</th><th>Pivot</th><th>Itn</th></tr>
<tr><td rowspan="3">1</td><td rowspan="3">(u_2, x_2)
$\langle v_2, y_2\rangle$</td><td>$(v_2, \bar{\theta})$</td><td>1</td></tr>
<tr><td>(u_2, x_2)</td><td>2</td></tr>
<tr><td rowspan="2">(u_3, y_2)</td><td rowspan="2">3</td></tr>
<tr><td rowspan="3">2</td><td rowspan="3">(u_3, x_1)
$\langle v_1, y_3\rangle$</td></tr>
<tr><td>(v_1, y_3)</td><td>4</td></tr>
<tr><td>$(\bar{\theta}, x_1)$</td><td>5</td></tr>
</table>

It is interesting to note how the third iteration of Lemke's method "links" the first and second iterations of the self-dual method.

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Dantzig has asserted that his self-dual parametric algorithm for solving a linear program is equivalent to Lemke's method for solving a linear complementary problem when the latter is applied to solve a linear program. ~~In~~ this paper, ~~we~~ formally prove that Dantzig's assertion is correct -- specifically that the point reached along the solution path after 2t iterations of Lemke's method is identical with the point reached after t iterations of Dantzig's method.

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